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Yang–Mills Fields and Gauge Gravity on Generalized Lagrange and Finsler Spaces

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Locally anisotropic gauge theories for semisimple and nonsemisimple groups are examined. A gauge approach to generalized Lagrange gravity based on local linear and affine structural groups is proposed.

1. INTRODUCTION

Despite the charm and success of general relativity there are some fundamental problems still unsolved in the framework of this theory. Here we point out the undetermined status of singularities, the problem of formulation of conservation laws in curved spaces, and the unrenormalizability of quantum gravitational interactions. To overcome these defects a number of authors (see, for example, Walner, 1985; Tseytlin, 1982; Luehr and Rosenbaum, 1980; Ponomarev *et al.*, 1985; Aldovandi and Stadile, 1984) tended to reconsider and reformulate gravitational theory as a gauge model similar to the theories of weak, electromagnetic, and strong forces. But, in spite of theoretical arguments and the attractive appearance of different proposed models of gauge gravity, the possibility and manner of interpretation of gravity as a kind of gauge interaction remain unclear.

The work of Popov and Daikhin (1975, 1976; Popov, 1975) is distinguished among other gauge approaches to gravity. Popov and Daikhin did not advance a gauge extension, or modification, of general relativity; they obtained an equivalent reformulation (such as well-known tetrad or spinoril variants) of the Einstein equations as Yang–Mills equations for correspondingly induced Cartan connections (Bishop and Crittenden, 1965) in the affine frame bundle on the pseudo-Riemannian space-time. This result was used in

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solving some specific problems in mathematical physics, for example, for formulation of a twistor-gauge interpretation of gravity and of nearly autoparallel conservation laws on curved spaces (Vacaru, 1987, 1993, 1994). It has also an important conceptual role. On one hand, it points to a possible unified treatment of gauge and gravitational fields in the language of linear connections in corresponding bundle spaces. On the other, it emphasizes that the two types of fundamental interactions mentioned essentially differ one from another, even if we admit for both of them a common gaugelike formalism, because if to Yang–Mills fields one associates semisimple gauge groups, to gauge treatments of Einstein gravitational fields one has to introduce into consideration nonsemisimple gauge groups.

Recent developments in theoretical physics suggest the idea that a more adequate description of radiational, statistical, and relativistic optic effects in classical and quantum gravity requires extensions of the geometrical backgrounds of theories (Vlasov, 1966; Vacaru *et al.*, 1994; Miron and Kawaguchi, 1991) by introducing into consideration spaces with local anisotropy and formulating corresponding variants of Lagrange and Finsler gravity (Miron and Anastasiei, 1993; Matsumoto, 1986; Asanov and Ponomarenko, 1989; Miron, 1985).

The aim of this work is twofold. The first objective is to formulate a geometrical approach to interactions of Yang–Mills fields on spaces with local anisotropy in the framework of the theory of linear connections in vector bundles (with semisimple structural groups) on generalized Lagrange spaces (Miron and Anastasiei, 1993). The second objective is to extend the geometrical formalism in a manner including theories with nonsemisimple groups which permit a unique fiber bundle treatment for both locally anisotropic Yang–Mills and gravitational interactions. In general lines, we shall follow the ideas and geometrical methods proposed in Tseytlin (1982), Ponomarev *et al.* (1985), Popov (1975), and Popov and Daikhin (1975, 1976), but we shall apply them in a form convenient for introducing into consideration geometrical constructions and physical theories on Lagrange and Finsler spaces.

One of the most important results of this paper is formulated as a theorem (see Section 5) stating that Miron's almost Hermitian Lagrange gravity (Miron and Anastasiei, 1993) is equivalent to a gaugelike theory in the bundle of affine adapted frames on generalized Lagrange spaces. This allows us a straightforward application of mathematical methods and computational techniques developed in the gauge field theories for construction of solutions of gravitational field equations describing gravitational gauge instantons with local anisotropy, and formulation and study of quantum and statistical models of physical interactions on curved, locally anisotropic spaces [first results are contained in Vacaru (1995a,b)].

The presentation is organized as follows:

Section 2 is a brief introduction in the Miron almost Hermitian model of generalized Lagrange geometry and corresponding extension of the Einstein theory. In Section 3 we give a geometrical interpretation of gauge (Yang–Mills) fields on generalized Lagrange spaces. Section 4 contains a geometrical definition of anisotropic Yang–Mills equations; the variational proof of gauge field equations is considered in connection with the "pure" geometrical method of introducing field equations. In Section 5 the generalized Lagrange gravity is equivalently reformulated as a gauge theory for nonsemisimple groups. A model of nonlinear de Sitter gauge gravity with local anisotropy is formulated in Section 6. We define gravitational gauge instantons with local anisotropy in Section 7. An outlook and conclusions are given in Section 8.

2. MIRON'S ALMOST HERMITIAN MODEL OF GENERALIZED LAGRANGE GEOMETRY

The goal of this section is to introduce the basic notations and definitions as well to present a brief review of the results on Lagrange and Finsler geometry necessary for our further considerations (Miron and Kawaguchi, 1991; Miron and Anastasiei, 1993; Matsumoto, 1986; Asanov and Ponomarenko, 1989; Miron, 1985).

Let M be a differentiable manifold of dimension n, $\dim M = m$, and TM its tangent bundle (differentiable means class C^∞ differentiability of functions). Local coordinates on open regions $U \subset M$ and $\mathcal{U} \subset TM$ are denoted, respectively, as $x = (x^i)$ and $u = (x, y) = u^\alpha = (x^i, y^j) = (x^i, y^{(j)})$, where Greek cumulative indices are used for values on TM and Latin indices $i, j, (i) = 1, 2, \ldots, n$ distinguish components of geometrical objects on base and fiber subspaces on TM.

A Lagrangian on M is a differentiable function $\mathcal{L}$: $TM \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the real number field, given locally as $\mathcal{L}$: $(x, y) \rightarrow \mathcal{L}(x, y)$ with the property that the tensorial distinguished field [d-field (Miron and Anastasiei, 1993)]

$$g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j} \tag{1}$$

is nondegenerate.

A pair $(M, \mathcal{L})$ forms a Lagrange space L^n with $g_{ij}(x, y)$ and $\mathcal{L}$ called, respectively, the fundamental tensor and fundamental function.

Remark 1. We obtain a Finsler space (M, L) as a particular case if $\mathcal{L} = L^2$, where L is the Finsler metric on M.

Miron (1985) and Miron and Anastasiei (1993) proposed to generalize the geometrical constructions in Finsler and Lagrange geometry by introducing into consideration arbitrary nonhomogeneous fundamental tensors $g_{ij}(x, y)$ not obligatorily generated by the fundamental function $\mathcal{L}$ (or L^2) as in (1).

Definition 1. A generalized Lagrange metric is a second-order covariant tensorial and nondegenerate d-field $g_{ij}(x, y)$ on M. The pair $M^n = (M, g_{ij}(x, y))$ is a generalized Lagrange space (GL-space).

We shall also use the tensorial d-field $g^{ij}(x, y)$ defined as satisfying the conditions

$$g_{ij}(x, y) g^{kj}(x, y) = \delta_j^k$$

In the total space of bundle TM we fix a nonlinear connection $N = \{N_j^i(x, y)\}$ defining a covariant derivation of vector $A = A^j s_j$ in direction $X = X^i \, \partial/\partial x^i$,

$$D_X A = X^i \left\{ \frac{\partial A^j}{\partial x^i} + N_i^j(x, A) \right\} S_j \tag{2}$$

where S_j is a basis of local linear sections of TM on M; components $N_i^j(x, A)$ are differentiable on x^i and A^j.

We shall consider decompositions of geometrical objects according to a locally adapted (to nonlinear connection N) basis denoted as

$$X_\alpha = (X_i, X_{(j)}) = \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^j} \right) \tag{3}$$

where

$$X_i = \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(u) \frac{\partial}{\partial y^j}, \qquad X_{(j)} = \frac{\delta}{\delta y^j} = \frac{\partial}{\partial y^j}$$

$\partial/\partial x^i$ and $\partial/\partial y^j$ are the usual partial derivations. The basis dual to (3)

$$X^\alpha = (X^i, X^{(j)}) = \delta u^\alpha = (\delta x^i, \delta y^j) \tag{4}$$

is defined by components $X^i = \delta x^i = dx^i$ and

$$X^{(j)} = \delta y^j = dy^j + N_i^j(u) \, dx^i$$

where dx^i and dy^j are, respectively, the usual differentials of variables x^i and y^j. The bases (3) and (4) are nonholonomic. We calculate nonholonomy coefficients by using commutations of vectors (3):

$$[X_\beta, X_\gamma] = X_\beta X_\gamma - X_\gamma X_\beta = w^\alpha{}_{\beta\gamma} X_\alpha \tag{5}$$

It is convenient to introduce in the total space of TM the linear normal d-connection compatible with the fundamental tensor on GL-space $L\Gamma(N) = (L^i_{jk}, C^i_{jk})$ with components induced by $N^a_i(x, y)$, $g_{ij}(x, y)$, and arbitrary given torsions $L^i_{\cdot jk} - L^i_{kj} = T^i_{\cdot jk}$, $C^i_{\cdot jk} - C^i_{\cdot kj} = S^i_{\cdot jk}$:

$$L^i_{\cdot jk}(x, y) = \frac{1}{2} g^{ir}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^k} + g_{rh}T^h_{\cdot jk} - g_{jh}T^h_{\cdot rk} + g_{kh}T^h_{\cdot jr}\right)$$

$$C^i_{\cdot jk}(x, y) = \frac{1}{2} g^{ir}\left(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r} + g_{rh}S^h_{\cdot jk} - g_{jh}S^h_{\cdot rk} + g_{kh}S^h_{\cdot jr}\right) \quad (6)$$

We point out that for a given nonlinear connection N we can naturally determine an almost complex structure F on TM (Miron and Anastasiei, 1993; Miron, 1985) with the property $F^2 = -I$, where

$$F(\delta/\delta x^i) = -\partial/\partial y^i, \qquad F(\partial/\partial y^i) = \delta/\delta x^i$$

Components $N^i_j(x, y)$ on TM uniquely induce a metric structure G on TM, called the N-lift on TM of the GL-metric $g_{ij}(x, y)$, defined by the equation

$$G\left(\frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right) = 0$$

or, in components, $G_{ij} - N^k_i G_{kj} = 0$.

With respect to the basis (4), the metric G on TM is written as

$$\begin{aligned} G(x, y) &= G_{\alpha\beta}(u^\gamma)\delta u^\alpha \otimes \delta u^\beta \\ &= g_{ij}(x, y)dx^i \otimes dx^j + g_{ij}(x, y)\delta y^i \otimes \delta y^j \end{aligned} \quad (7)$$

Definition 2 (Miron, 1985; Miron and Anastasiei, 1993). The space $H^{2n} = (TM, G, F)$ is called the almost Hermitian N-model of the GL-space $M^n = (M, g_{ij}(x, y))$ endowed with nonlinear connection $N^k_i(x, y)$.

By $\overset{N}{\nabla}$ we denote the almost Hermitian linear covariant derivation [the N-lift on TM of the $L\Gamma(N)$ connection (6)] with coefficients $\overset{N}{\Gamma}{}^\alpha{}_{\beta\gamma}$ defined by

$$\overset{N}{\nabla}_{X_\alpha} X_\beta = \overset{N}{\Gamma}{}^\alpha{}_{\beta\gamma} X_\alpha \quad (8)$$

Using local adapted bases (3) and (4), we can distinguish components of connection $\overset{N}{\Gamma}{}^\alpha{}_{\beta\gamma}$ and nonholonomy coefficients $w^\alpha{}_{\beta\gamma}$ from (5):

$$\begin{aligned} &\overset{N}{\Gamma}{}^i_{jk} = L^i_{jk}, \quad &&\overset{N}{\Gamma}{}^i{}_{(j)k} = 0, \quad &&\overset{N}{\Gamma}{}^i{}_{j(k)} = 0, \quad &&\overset{N}{\Gamma}{}^i{}_{(j)(k)} = 0 \\ &\overset{N}{\Gamma}{}^{(i)}_{jk} = 0, \quad &&\overset{N}{\Gamma}{}^{(i)}{}_{(j)k} = 0, \quad &&\overset{N}{\Gamma}{}^{(i)}{}_{j(k)} = 0, \quad &&\overset{N}{\Gamma}{}^{(i)}_{(j)(k)} = C^i_{jk} \end{aligned} \quad (9)$$

and, respectively,

$$w^{i}_{jk} = 0, \qquad w^{i}_{(j)k} = 0, \qquad w^{i}_{j(k)} = 0, \qquad w^{i}_{(j)(k)} = 0, \qquad w^{(i)}_{(j)k} = 0$$

$$w^{(i)}_{jk} = R^{i}_{jk} = \frac{\delta N^{i}_{j}}{\delta x^{k}} - \frac{\delta N^{i}_{k}}{\delta x^{i}}, \qquad w^{(i)}_{(j)k} = -\frac{\partial N^{i}_{k}}{\partial y^{j}}, \qquad w^{(i)}_{(j)(k)} = \frac{\partial N^{i}_{j}}{\partial y^{k}} \tag{10}$$

We point out that the metric (7) is compatible with the $\overset{N}{\nabla}$-connection, i.e., $\overset{N}{\nabla}_{\alpha} G_{\beta\gamma} = 0$.

The torsion $\overset{N}{T}(X_{\alpha}, X_{\beta}) = \overset{N}{T}{}^{\alpha}{}_{\beta\gamma} X_{\alpha}$ and the curvature $\overset{N}{R}(X_{\delta}, X_{\gamma})X_{\beta} = \overset{N}{R}_{\beta}{}^{\alpha}{}_{\gamma\delta} X_{\alpha}$ of the connection $\overset{N}{\nabla}$ are defined in the usual manner:

$$\overset{N}{T}{}^{\alpha}{}_{\beta\gamma} = \overset{N}{\Gamma}{}^{\alpha}{}_{\beta\gamma} - \overset{N}{\Gamma}{}^{\alpha}{}_{\beta\gamma} + w^{\alpha}{}_{\beta\gamma} \tag{11}$$

and, respectively,

$$\overset{N}{R}{}_{\beta\cdot\gamma\delta}^{\cdot\alpha} = X_{\delta} \overset{N}{\Gamma}{}^{\alpha}{}_{\beta\gamma} - X_{\gamma} \overset{N}{\Gamma}{}^{\alpha}{}_{\beta\delta} + \overset{N}{\Gamma}{}^{\cdot\varphi}_{\beta\gamma} \overset{N}{\Gamma}{}^{\alpha}{}_{\cdot\varphi\delta} - \overset{N}{\Gamma}{}^{\cdot\varphi}_{\beta\delta} \overset{N}{\Gamma}{}^{\alpha}{}_{\varphi\gamma} + \overset{N}{\Gamma}{}^{\alpha}{}_{\beta\varphi} w^{\varphi}_{\gamma\delta} \tag{12}$$

Now, introducing the Ricci, $\overset{N}{R}_{\alpha\beta} = \overset{N}{R}{}_{\alpha\cdot\beta\gamma}^{\cdot\gamma}$, and scalar curvature, $\overset{N}{R} = G^{\alpha\beta} \overset{N}{R}_{\alpha\beta}$, fields, we can write the Einstein equations on H^{2n}-spaces:

$$\overset{N}{R}_{\alpha\beta} - \frac{1}{2} G_{\alpha\beta} \overset{N}{R} = \kappa \overset{N}{T}_{\alpha\beta} \tag{13}$$

where κ is the gravitational constant and $\overset{N}{T}_{\alpha\beta}$ is the H^{2n}-energy-momentum tensor of matter. Here we emphasize that $\overset{N}{T}_{\alpha\beta}$ do not satisfy the conservation law

$$\overset{N}{\nabla}_{\alpha} \overset{N}{T}{}^{\alpha}{}_{\beta} = 0$$

because

$$\overset{N}{\nabla}_{\alpha} \left(\overset{N}{R}{}^{\alpha}_{\beta} - \frac{1}{2} \overset{N}{R} \delta^{\alpha}_{\beta} \right) \neq 0$$

H^{2n}-space-time will be considered as the base space for our fiber bundle approach to locally anisotropic gauge and gravitational interactions.

3. GAUGE FIELDS ON H^{2n}-SPACES

This section is devoted to formulation of the geometrical background for gauge field theories on spaces with local anisotropy.

Let $(P, \pi, \mathrm{Gr}, H^{2n})$ be a principal bundle on base H^{2n} with structural group Gr and surjective map $\pi: P \to H^{2n}$. At every point $u = (x, y) \in H^{2n}$ there is a vicinity $\mathcal{U} \subset H^{2n}$, $u \in \mathcal{U}$, with trivializing P diffeomorphisms f and φ:

$$f_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathrm{Gr}, \qquad f(p) = (\pi(p), \varphi(p))$$

$$\varphi_{\mathcal{U}}: \pi^{-1}(\mathcal{U}) \to \mathrm{Gr}, \qquad \varphi(pq) = \varphi(p)q, \qquad \forall q \in \mathrm{Gr}, \quad p \in P$$

We remark that in the general case for two open regions

$$\mathcal{U}, \mathcal{V} \subset H^{2n}, \mathcal{U} \cap \mathcal{V} \neq \varnothing, f_{\mathcal{U}|p} \neq f_{\mathcal{V}|p}, \text{ even } p \in \mathcal{U} \cap \mathcal{V}.$$

Transition functions $g_{\mathcal{U}\mathcal{V}}$ are defined as

$$g_{\mathcal{U}\mathcal{V}}: \mathcal{U} \cap \mathcal{V} \to \mathrm{Gr}, \qquad g_{\mathcal{U}\mathcal{V}}(u) = \varphi_{\mathcal{U}}(p)(\varphi_{\mathcal{V}}(p))^{-1}, \qquad \pi(p) = u$$

Trivialization generates local sections:

$$s_{\mathcal{U}}: \mathcal{U} \to \pi^{-1}(\mathcal{U}), \qquad s_{\mathcal{U}}(u) = p\varphi_{\mathcal{U}}(p)^{-1}, \qquad \pi(p) = u$$

Hereafter we shall omit, for simplicity, the specification of trivializing regions of maps and denote, for example, $f \equiv f_{\mathcal{U}}$, $\varphi \equiv \varphi_{\mathcal{U}}$, $s \equiv s_{\mathcal{U}}$, if this will not give rise to ambiguities.

Let θ be the canonical left invariant 1-form on Gr with values in algebra Lie $\mathcal{G}$ of group Gr uniquely defined from the relation $\theta(q) = q$, $\forall q \in \mathcal{G}$, and consider a 1-form ω on $\mathcal{U} \subset H^{2n}$ with values in $\mathcal{G}$. Using θ and ω, we can locally define the connection form Ω in P as a 1-form:

$$\Omega = \varphi^*\theta + \mathrm{Ad}\, \varphi^{-1}(\pi^*\omega) \tag{14}$$

where $\varphi^*\theta$ and $\pi^*\omega$ are, respectively, forms induced on $\pi^{-1}(\mathcal{U})$ and P by maps φ and π and $\omega = s^*\Omega$. The adjoint action on a form λ with values in $\mathcal{G}$ is defined as

$$(\mathrm{Ad}\, \varphi^{-1}\lambda)_p = (\mathrm{Ad}\, \varphi^{-1}(p))\lambda_p$$

where λ_p is the value of form λ at point $p \in P$.

Introducing a basis $\{\Delta_{\hat{a}}\}$ in $\mathcal{G}$ (index $\hat{a}$ enumerates the generators making up this basis), we write the 1-form ω on H^{2n} as

$$\omega = \Delta_{\hat{a}}\omega^{\hat{a}}(u), \qquad \omega^{\hat{a}}(u) = \omega^{\hat{a}}_{\mu}(u)\delta u^{\mu} \tag{15}$$

where $\delta u^{\mu} = (\delta x^i, \delta y^j)$ is a local adapted basis on H^{2n} and the Einstein summation rule on indices $\hat{a}$ and μ is used. Functions $\omega^{\hat{a}}_{\mu}(u) = \omega^{\hat{a}}_{\mu}(x, y)$ from (15) will be called the components of Yang–Mills fields on H^{2n}-space. Gauge transforms of ω can be geometrically interpreted as transition relations for $\omega_{\mathcal{U}}$ and $\omega_{\mathcal{V}}$, when $u \in \mathcal{U} \cap \mathcal{V}$,

$$(\omega_{\mathcal{U}})_u = (g^*_{\mathcal{U}\mathcal{V}}\theta)_u + \mathrm{Ad}\, g_{\mathcal{U}\mathcal{V}}(u)^{-1}(\omega_{\mathcal{V}})_u \tag{16}$$

To relate $\omega_\mu^{\hat{a}}$ with a covariant derivation we shall consider a vector bundle E associated to P. Let ρ: Gr $\to GL(\mathbb{R}^m)$ and ρ': $\mathcal{G} \to \mathrm{End}(E^m)$ be, respectively, linear representations of group Gr and Lie algebra $\mathcal{G}$ (in a more general case we can consider $\mathbb{C}^m$ instead of $\mathbb{R}^m$). Map ρ defines a left action on Gr and associated vector bundle $E = P \times \mathbb{R}^m/\mathrm{Gr}$, π_E: $E \to H^{2n}$. Introducing the standard basis $\xi_{\underline{i}} = \{\xi_{\underline{1}}, \xi_{\underline{2}}, \ldots, \xi_{\underline{m}}\}$ in $\mathbb{R}^m$, we can define the right action on $P \times \mathbb{R}^m$, $((p, \xi)q = (pq, \rho(q^{-1})\xi)$, $q \in \mathrm{Gr})$, the map induced from P

$$p: \mathbb{R}^m \to \pi_E^{-1}(u), \qquad (p(\xi) = (p\xi)\mathrm{Gr},\ \xi \in \mathbb{R}^m,\ \pi(p) = u)$$

and a basis of local sections $e_{\underline{i}}$: $U \to \pi_E^{-1}(U)$, $e_{\underline{i}}(u) = s(u)\xi_{\underline{i}}$. Every section ζ: $H^{2n} \to E$ can be written locally as $\zeta = \zeta^i e_i$, $\zeta^i \in C^\infty(\mathcal{U})$. To every vector field X on H^{2n} and Yang–Mills field $\omega^{\hat{a}}$ on P we associate operators of covariant derivations:

$$\nabla_X\zeta = e_{\underline{i}}[X\zeta^{\underline{i}} + B(X)^{\underline{i}}_{\underline{j}}\zeta^{\underline{j}}]$$

$$B(X) = (\rho' X)_{\hat{a}}\omega^{\hat{a}}(X) \tag{17}$$

Transformation laws (16) and operators (17) are interrelated by these transition transforms for values $e_{\underline{i}}$, $\zeta^{\underline{i}}$, and B_μ:

$$e_{\underline{i}}^{\mathcal{V}}(u) = [\rho g_{\mathcal{U}\mathcal{V}}(u)]_{\underline{i}}^{\underline{j}} e_{\underline{j}}^{\mathcal{U}}$$

$$\zeta_{\mathcal{U}}^{\underline{i}}(u) = [\rho g_{\mathcal{U}\mathcal{V}}(u)]_{\underline{j}}^{\underline{i}} \zeta_{\mathcal{V}}^{\underline{j}}$$

$$B_\mu^{\mathcal{V}}(u) = [\rho g_{\mathcal{U}\mathcal{V}}(u)]^{-1}\delta_\mu[\rho g_{\mathcal{U}\mathcal{V}}(u)] + [\rho g_{\mathcal{U}\mathcal{V}}(u)]^{-1} B_\mu^{\mathcal{U}}(u)[\rho g_{\mathcal{U}\mathcal{V}}(u)] \tag{18}$$

where $B_\mu^{\mathcal{U}}(u) = B^\mu(\delta/\delta u^\mu)(u)$.

Using (18), we can verify that the operator $\nabla_X^{\mathcal{U}}$, acting on sections of π_E: $E \to H^{2n}$ according to definition (17), satisfies the properties

$$\nabla^{\mathcal{U}}_{f_1X+f_2Y} = f_1\nabla_X^{\mathcal{U}} + f_2\nabla_Y^{\mathcal{U}}$$

$$\nabla_X^{\mathcal{U}}(f\zeta) = f\nabla_X^{\mathcal{U}}\zeta + (Xf)\zeta$$

$$\nabla_X^{\mathcal{U}}\zeta = \nabla_X^{\mathcal{V}}\zeta, \qquad u \in \mathcal{U} \cap \mathcal{V}, \quad f_1, f_2 \in C^\infty(\mathcal{U})$$

So, we can conclude that the Yang–Mills connection in the vector bundle π_E: $E \to H^{2n}$ is not a general one, but is induced from the principal bundle π: $P \to H^{2n}$ with structural group Gr.

The curvature $\mathcal{F}$ of the connection Ω from (14) is defined as

$$\mathcal{F} = D\Omega, \qquad D = \hat{H} \circ d \tag{19}$$

where d is the operator of exterior derivation acting on $\mathcal{G}$-valued forms as

$d(\Delta_{\hat{a}} \otimes \chi^{\hat{a}}) = \Delta_{\hat{a}} \otimes d\chi^{\hat{a}}$ and $\hat{H}$ is the horizontal projecting operator acting, for example, on the 1-form λ as $(\hat{H}\lambda)_P(X_p) = \lambda_p(H_p X_p)$, where H_p projects on the horizontal subspace $\mathcal{H}_p \in P_p$ [$X_p \in \mathcal{H}_p$ is equivalent to $\Omega_p(X_p) = 0$]. We can express (19) locally as

$$\mathcal{F} = \text{Ad}\ \varphi_{\mathcal{U}}^{-1}(\pi^* \mathcal{F}_{\mathcal{U}}) \tag{20}$$

where

$$\mathcal{F}_{\mathcal{U}} = d\omega_{\mathcal{U}} + \frac{1}{2}[\omega_{\mathcal{U}}, \omega_{\mathcal{U}}] \tag{21}$$

The exterior product of $\mathcal{G}$-valued forms from (21) is defined as

$$[\Delta_{\hat{a}} \otimes \lambda^{\hat{a}}, \Delta_{\hat{b}} \otimes \xi^{\hat{b}}] = [\Delta_{\hat{a}}, \Delta_{\hat{b}}] \otimes \lambda^{\hat{a}} \wedge \lambda^{\hat{b}}$$

where the antisymmetric tensorial product is

$$\lambda^{\hat{a}} \wedge \xi^{\hat{b}} = \lambda^{\hat{a}}\xi^{\hat{b}} - \xi^{\hat{b}}\lambda^{\hat{a}}$$

Introducing structural coefficients $f_{\hat{b}\hat{c}}{}^{\hat{a}}$ of $\mathcal{G}$ satisfying

$$[\Delta_{\hat{b}}, \Delta_{\hat{c}}] = f_{\hat{b}\hat{c}}{}^{\hat{a}}\Delta_{\hat{a}}$$

we can rewrite (21) in a form more convenient for local considerations:

$$\mathcal{F}_{\mathcal{U}} = \Delta_{\hat{a}} \otimes \mathcal{F}^{\hat{a}}{}_{\mu\nu}\delta u^{\mu} \wedge \delta u^{\nu} \tag{22}$$

where

$$\mathcal{F}^{\hat{a}}{}_{\mu\nu} = \frac{\delta\omega_{\nu}^{\hat{a}}}{\delta u^{\mu}} - \frac{\delta\omega^{\hat{a}}{}_{\mu}}{\delta u^{\nu}} + \frac{1}{2} f_{\hat{b}\hat{c}}{}^{\hat{a}}(\omega_{\mu}^{\hat{b}}\omega_{\nu}^{\hat{c}} - \omega_{\nu}^{\hat{b}}\omega_{\mu}^{\hat{c}})$$

This section ends by considering the problem of reduction of the local anisotropic gauge symmetries and gauge fields to isotropic ones. For local trivial considerations we can consider that the vanishing of dependences on y variables leads to isotropic Yang–Mills fields with the same gauge group as in the anisotropic case. Global geometric constructions require a more rigorous topological study of possible obstacles for reduction of total spaces and structural groups on anisotropic bases to their analogs on isotropic (for example, pseudo-Riemannian) base spaces.

4. YANG–MILLS EQUATIONS FOR GAUGE FIELDS ON H^{2n}-SPACES

Interior gauge (nongravitational) symmetries are associated to semisimple structural groups. On the principal bundle $(P, \pi, \text{Gr}, H^{2n})$ with nondegenerate Killing form for the semisimple group Gr we can define the generalized Lagrange metric

$$h_p(X_p, Y_P) = G_{\pi(p)}(d\pi_P X_P, d\pi_P Y_P) + K(\Omega_P(X_P), \Omega_P(X_P)) \qquad (23)$$

where $d\pi_P$ is the differential of map π: $P \to H^{2n}$, $G_{\pi(p)}$ is locally generated as the H^{2n}-metric (7), and K is the Killing form on $\mathcal{G}$:

$$K(\Delta_{\hat{a}}, \Delta_{\hat{b}}) = f_{\hat{b}\hat{d}}{}^{\hat{c}} f_{\hat{a}\hat{c}}{}^{\hat{d}} = K_{\hat{a}\hat{b}}$$

Using the metric $G_{\alpha\beta}$ on H^{2n} [$h_P(X_P, Y_P)$ on P], we can introduce operators $*_G$ and $\hat{\delta}_G$ acting in the space of forms on H^{2n} ($*_h$ and $\hat{\delta}_h$ acting on forms on H^{2n} with values in $\mathcal{G}$). Let $e_{\underline{\mu}}$ be orthonormalized frames on $\mathcal{U} \subset H^{2n}$ and e^{μ} the adjoint coframes. Locally

$$G = \sum_{\mu} \eta(\mu) e^{\mu} \otimes e^{\mu}$$

where $\eta_{\mu\mu} = \eta(\mu) = \pm 1$, $\mu = 1, 2, \ldots, 2n$, and the Hodge operator $*_G$ can be defined as $*_G$: $\Lambda^r(H^{2n}) \to \Lambda^{2n-r}(H^{2n})$, or, in explicit form, as

$$\begin{aligned} *_G(e^{\mu_1} \wedge \cdots \wedge e^{\mu_r}) = {} & \eta(\nu_1) \cdots \eta(\nu_{2n-r}) \\ & \times \operatorname{sign}\begin{pmatrix} 1 & 2 & \cdots & & \cdots & 2n \\ \mu_1 & \mu_2 & \cdots & \mu_r \nu_1 & \cdots & \nu_{2n-r} \end{pmatrix} \\ & \times e^{\nu_1} \wedge \cdots \wedge e^{\nu_{2n-r}} \end{aligned} \qquad (24)$$

Next, define the operator

$$*_G^{-1} = \eta(1) \cdots \eta(2n)(-1)^{r(2n-r)} *_G$$

and introduce the scalar product on forms $\beta_1, \beta_2, \ldots \subset \Lambda^r(H^{2n})$ with compact carrier:

$$(\beta_1, \beta_2) = \eta(1) \cdots \eta(2n) \int \beta_1 \wedge *_G \beta_2$$

The operator $\hat{\delta}_G$ is defined as the adjoint to d associated to the scalar product for forms, specified for r-forms as

$$\hat{\delta}_G = (-1)^r *_G^{-1} \circ d \circ *_G \qquad (25)$$

We remark that operators $*_h$ and δ_h acting in the total space of P can be defined similarly to (24) and (25), but by using metric (23). Both these operators also act in the space of $\mathcal{G}$-valued forms:

$$*(\Delta_{\hat{a}} \otimes \varphi^{\hat{a}}) = \Delta_{\hat{a}} \otimes (*\varphi^{\hat{a}})$$

$$\hat{\delta}(\Delta_{\hat{a}} \otimes \varphi^{\hat{a}}) = \Delta_{\hat{a}} \otimes \hat{\delta}\varphi^{\hat{a}}$$

The form λ on P with values in $\mathcal{G}$ is called (1) horizontal if $\hat{H}\lambda = \lambda$ and (2) equivariant if $R^*(q)\lambda = \text{Ad } q^{-1}\varphi$, $\forall q \in \text{Gr}$, $R(q)$ being the right shift on P. We can verify that equivariant and horizontal forms also satisfy the conditions

$$\lambda = \text{Ad } \varphi_{\mathcal{U}}^{-1}(\pi^*\lambda), \qquad \lambda_{\mathcal{U}} = \overset{*}{S}_{\mathcal{U}}\lambda$$

$$(\lambda_{\mathcal{V}})_{\mathcal{U}} = \text{Ad}(g_{\mathcal{U}\mathcal{V}}(u))^{-1}(\lambda_{\mathcal{U}})_u$$

Now, we can define the field equations for curvature (20) and connection (14):

$$\Delta\mathcal{F} = 0 \tag{26}$$

$$\nabla\mathcal{F} = 0 \tag{27}$$

where $\Delta = \hat{H} \circ \hat{\delta}_h$. Equations (26) are similar to the well-known Maxwell equations and for non-Abelian gauge fields and are called Yang–Mills equations. The structural equations (27) are called Bianchi identities.

The field equations (26) do not have a physical meaning because they are written in the total space of bundle E and not on the base anisotropic space-time H^{2n}. But this difficulty may be obviated by projecting the mentioned equations on the base. The 1-form $\Delta\mathcal{F}$ is horizontal by definition and its equivariance follows from the right invariance of metric (23). So, there is a unique form $(\Delta\mathcal{F})_{\mathcal{U}}$ satisfying

$$\Delta\mathcal{F} = \text{Ad } \varphi_{\mathcal{U}}^{-1}\pi^*(\Delta\mathcal{F})_{\mathcal{U}}$$

Projection of (26) on the base can be written as $(\Delta\mathcal{F})_{\mathcal{U}} = 0$. To calculate $(\Delta\mathcal{F})_{\mathcal{U}}$, we use the equality (Bishop and Crittenden, 1965; Popov and Daikhin, 1976)

$$d(\text{Ad } \varphi_{\mathcal{U}}^{-1}\lambda) = \text{Ad } \varphi_{\mathcal{U}}^{-1}\, d\lambda - [\overset{*}{\varphi}_{\mathcal{U}}\theta, \text{Ad } \varphi_{\mathcal{U}}^{-1}\lambda] \tag{28}$$

where λ is a form on P with values in $\mathcal{G}$. For r-forms we have

$$\hat{\delta}(\text{Ad } \varphi_{\mathcal{U}}^{-1}\lambda) = \text{Ad } \varphi_{\mathcal{U}}^{-1}\hat{\delta}\lambda - (-1)^r *_h[\overset{*}{\varphi}_{\mathcal{U}}\theta, *_h \text{Ad } \varphi_{\mathcal{U}}^{-1}\lambda]$$

and, as a consequence,

$$\hat{\delta}\mathcal{F} = \text{Ad } \varphi_{\mathcal{U}}^{-1}\{\hat{\delta}_h\pi^*\mathcal{F}_{\mathcal{U}} + *_h^{-1}[\pi^*\omega_{\mathcal{U}}, *_h\pi^*\mathcal{F}_{\mathcal{U}}]\}$$

$$- *_h^{-1}[\Omega, \text{Ad } \varphi_{\mathcal{U}}^{-1} *_h(\pi^*\mathcal{F})] \tag{29}$$

By using straightforward calculations in the adapted dual basis on $\pi^{-1}(\mathcal{U})$ we can verify the equalities

$$[\Omega, \text{Ad } \varphi_{\mathcal{U}}^{-1} *_h(\pi^*\mathcal{F}_{\mathcal{U}})] = 0, \qquad \hat{H}\delta_h(\pi^*\mathcal{F}_{\mathcal{U}}) = \pi^*(\hat{\delta}_G\mathcal{F})$$

$$*_h^{-1}[\pi^*\omega_{\mathcal{U}}, *_h(\pi^*\mathcal{F}_{\mathcal{U}})] = \pi^*\{*_G^{-1}[\omega_{\mathcal{U}}, *_G\mathcal{F}_{\mathcal{U}}]\} \tag{30}$$

From (29) and (30) it follows that

$$(\Delta\mathcal{F})_{\mathcal{U}} = \hat{\delta}_G\mathcal{F}_{\mathcal{U}} + *_{G^{-1}}[\omega_{\mathcal{U}}, *_G\mathcal{F}] \tag{31}$$

Taking into account (31) and (25), we prove that projection on H^{2n} of equations (26) and (27) can be expressed respectively as

$$*_G^{-1} \circ d \circ *_G\mathcal{F}_{\mathcal{U}} + *_G^{-1}[\omega_{\mathcal{U}}, *_G\mathcal{F}_{\mathcal{U}}] = 0 \tag{32}$$

$$d\mathcal{F}_{\mathcal{U}} + [\omega_{\mathcal{U}}, \mathcal{F}_{\mathcal{U}}] = 0 \tag{33}$$

Equations (32) [see (31)] are gauge-invariant because

$$(\Delta\mathcal{F})_{\mathcal{U}} = \mathrm{Ad}\; g_{\mathcal{U}\mathcal{V}}^{-1}(\Delta\mathcal{F})_{\mathcal{V}}$$

By using formulas (22)–(25) we can rewrite (32) in coordinate form

$$\overset{N}{\nabla}_\nu(G^{\nu\lambda}\mathcal{F}^{\hat{a}}{}_{\lambda\mu}) + f_{\hat{b}\hat{c}}{}^{\hat{a}}G^{\nu\lambda}\omega_\lambda{}^b\mathcal{F}_{\nu\mu} = 0 \tag{34}$$

where $\overset{N}{\nabla}_\nu$ is the covariant derivation on H^{2n}-space [see (8)].

It is possible to distinguish the x and y parts of equations (34) by using formulas (7)–(19).

We point out that for bundles with semisimple structural groups the Yang–Mills equations (26) [and, as a consequence, their horizontal projections (32) or (34)] can be obtained by variation of the action

$$I = \int \mathcal{F}^{\hat{a}}{}_{\mu\nu}\mathcal{F}^{\hat{b}}{}_{\alpha\beta}G^{\mu\alpha}G^{\nu\beta}K_{\hat{a}\hat{b}}|K_{\alpha\beta}|^{1/2}\, dx^1 \cdots dx^n\delta y^1 \cdots \delta y^n \tag{35}$$

Equations for extremals of (35) have the form

$$K_{\hat{r}\hat{b}}G^{\lambda\alpha}G^{\kappa\beta}\overset{N}{\nabla}_\alpha\mathcal{F}^{\hat{b}}{}_{\lambda\beta} - K_{\hat{a}\hat{b}}G^{\kappa\alpha}G^{\nu\beta}f_{\hat{r}\hat{l}}{}^{\hat{a}}\omega_\nu{}^{\hat{l}}\mathcal{F}^{\hat{b}}{}_{\alpha\beta} = 0 \tag{36}$$

which are equivalent to "pure" geometric equations (34) [or (32)] due to nondegeneration of the Killing form $K_{\hat{r}\hat{b}}$ for semisimple groups.

To take into account gauge interactions with matter fields (section of vector bundle **E** on H^{2n}) we have to introduce a source 1-form $\mathcal{J}$ in equations (26) and to write them as

$$\Delta\mathcal{F} = \mathcal{J} \tag{37}$$

Explicit constructions of $\mathcal{J}$ require concrete definitions of the bundle E; for example, for spinoril fields an invariant formulation of the Dirac equations on H^{2n}-spaces is necessary. We omit spinoril considerations in this work, but we shall present the definition of the source $\mathcal{J}$ for gravitational interactions (by using the energy-momentum tensor of matter on H^{2n}-space) in the next section.

5. GENERALIZED LAGRANGE GRAVITY AS A GAUGE THEORY FOR NONSEMISIMPLE GROUPS

A considerable body of work on the formulation of gauge gravitational models on isotropic spaces is based on using nonsemisimple groups, for example, Poincaré and affine groups, as structural gauge groups [see critical analysis and original results in Walner (1985), Tseytlin (1982), Luehr and Rosenbaum (1980), and Ponomarev *et al.* (1985)]. The main impediment to developing such models is caused by the degeneration of Killing forms for nonsemisimple groups, which make it impossible to construct consistent variational gauge field theories [functional, (35), and extremal equations, (36), are degenerate in these cases]. There are at least two possibilities to get around the mentioned difficulty. The first is to realize a minimal extension of the nonsemisimple group to a semisimple one, similar to the extension of the Poincaré group to the de Sitter group considered in Tseytlin (1982), Ponomarev (1985), Popov and Daikhin (1975) and Asanov and Ponomarenko (1989) (in the next section we shall use this operation for the definition of anisotropic gravitational instantons). The second possibility is to introduce into consideration the bundle of adapted affine frames on H^{2n}, to use an auxiliary nondegenerate bilinear form $a_{\hat{a}\hat{b}}$ instead of the degenerate Killing form $K_{\hat{a}\hat{b}}$, and to consider a "pure" geometric method, illustrated in the previous section, of defining gauge field equations. Projecting on the base H^{2n}, we shall obtain gauge gravitational field equations on GL-space having a form similar to Yang–Mills equations.

The goal of this section is to prove that a specific parametrization of components of the Cartan connection (1) in the bundle of adapted affine frames on H^{2n} establishes an equivalence between Yang–Mills equations (37) and Einstein equations (13) on H^{2n}-spaces.

5.1. The Bundle of Adapted Linear Frames on H^{2n}: $\mathcal{L}_a = (La(H^{2n}), GL_{n+n}(\mathbb{R}), H^{2n})$

Let $(X_\alpha)_u = (X_i, X_{(j)})_u$ be an adapted frame [see (3) and (4)] at point $u \in H^{2n}$. We consider a local right distinguished action of matrices

$$A_{\alpha'}{}^{\alpha} = \begin{pmatrix} A_{i'}{}^{i} & 0 \\ 0 & B_{j'}{}^{j} \end{pmatrix} \subset GL_{n+n} = GL(n, \mathbb{R}) \oplus GL(n, \mathbb{R})$$

Nondegenerate matrices $A_{i'}{}^{i}$ and $B_{j'}{}^{j}$ respectively transform linearly $X_{i|u}$ into $X_{i'|u} = A_{i'}{}^{i}X_{i|u}$ and $X_{(j')|u}$ into $X_{(j')|u} = B_{j'}{}^{j}X_{(j)|u}$, where $X_{\alpha'|u} = A_{\alpha'}{}^{\alpha}X_{\alpha|u}$ is also an adapted frame at the same point $u \in H^{2n}$. We denote by $La(H^{2n})$ the set of all adapted frames X_α at all points of H^{2n} and consider the surjective map π from $La(H^{2n})$ to H^{2n} transforming every adapted frame $X_{\alpha|u}$ and point

u into point u. Every $X_{\alpha'|u}$ has a unique representation as $X_{\alpha'} = A_{\alpha'}{}^{\alpha} X_{\alpha}{}^{(0)}$, where $X^{(0)}_{\alpha}$ is a fixed distinguished basis in tangent space $T(H^{2n})$. It is obvious that $\pi^{-1}(\mathcal{U})$, $\mathcal{U} \subset H^{2n}$, is bijective to $\mathcal{U} \times GL_{n+n}(\mathbb{R})$. We can transform $La(H^{2n})$ in a differentiable manifold taking $(u^{\beta}, A_{\alpha'}{}^{\alpha})$ as a local coordinate system on $\pi^{-1}(\mathcal{U})$. Now, it is easy to verify that $\mathcal{L}a(H^{2n}) = (La(H^{2n}, H^{2n}, GL_{n+n}(\mathbb{R}))$ is a principal bundle. We call $\mathcal{L}a(H^{2n})$ the bundle of linear adapted frames on H^{2n}.

The next step is to identify the components of connection (14) in $\mathcal{L}a(H^{2n})$ projected on base H^{2n} with components of the almost Hermitian connection $\overset{N}{\Gamma}{}^{\alpha}{}_{\lambda\beta}$ [see (8) and (9)]:

$$\Omega^{\hat{a}}_{\mathcal{U}} = \omega^{\hat{a}} = \{\omega^{\hat{\alpha}\hat{\beta}}\lambda := \overset{N}{\Gamma}{}^{\alpha}{}_{\lambda\beta}\} \tag{38}$$

Introducing (38) in (31), we calculate the local 1-form

$$(\Delta\mathcal{R})_{\mathcal{U}} = \Delta_{\hat{\alpha}\hat{\alpha}_1} \otimes (G^{\nu\lambda}\overset{N}{\nabla}_{\lambda}\mathcal{R}^{\hat{\alpha}\hat{\alpha}_1}{}_{\nu\mu} + f^{\hat{\alpha}\hat{\alpha}_1}{}_{\hat{\beta}\hat{\beta}_1\hat{\gamma}\hat{\gamma}_1} G^{\nu\lambda}\omega^{\hat{\beta}\hat{\beta}_1}{}_{\lambda}\, \mathcal{R}^{\hat{\gamma}\hat{\gamma}_1}{}_{\nu\mu})\delta u^{\mu} \tag{39}$$

where

$$\Delta_{\hat{\alpha}\hat{\alpha}_1} = \begin{pmatrix} \Delta_{ii_1} & 0 \\ 0 & \Delta_{(j)(j_1)} \end{pmatrix}$$

is the standard distinguished basis in Lie algebra of matrices $\mathcal{G}l_{n+n}(\mathbb{R})$ with $(A_{ii_1})_{aa_1} = \delta_{ia}\delta_{i_1a_1}$ and $(\Delta_{(i)(j_1)})_{bb_1} = \delta_{(j)b}\delta_{(j_1)b_1}$ being the standard basis in $\mathcal{G}l(\mathbb{R}^n)$. We have denoted the curvature of connection (38), considered in (39), as

$$\mathcal{R}_{\mathcal{U}} = \Delta_{\hat{\alpha}\hat{\alpha}_1} \otimes \mathcal{R}^{\hat{\alpha}\hat{\alpha}_1}{}_{\mu\nu} X^{\mu} \wedge X^{\nu} \tag{40}$$

where $\mathcal{R}^{\hat{\alpha}\hat{\alpha}_1}{}_{\mu\nu} = \overset{N}{R}{}_{\alpha_1}{}^{\alpha}{}_{\mu\nu}$ [see the almost Hermitian curvature (12)].

5.2. The Bundle of Adapted Affine Frames on H^{2n}: $\mathcal{A}a(H^{2n}) = (Aa(H^{2n}), Af_{n+n}(\mathbb{R}), H^{2n})$

Besides $\mathcal{L}a(H^{2n})$ with GL-space H^{2n}, another bundle is naturally related, the bundle of adapted affine frames with structural group $Af_{n+n}(\mathbb{R})$ being a semidirect product of $GL_{n+n}(\mathbb{R})$ and distinguished $\mathbb{R}^{n+n}$, $Af_{n+n}(\mathbb{R}) = GL_{n+n}(\mathbb{R}) \otimes \mathbb{R}^{n+n}$. Because as linear space the Lie algebra $af_{n+n}(\mathbb{R})$ is a direct sum of $\mathcal{G}l_{n+n}(\mathbb{R})$ and $\mathbb{R}^{n+n}$, we can write forms on $\mathcal{A}a(H^{2n})$ as $\Theta = (\Theta_1, \Theta_2)$, where Θ_1 is the $\mathcal{G}l_{n+n}(\mathbb{R})$ component and Θ_2 is the $\mathbb{R}^{n+n}$ component of the form Θ. Connection (38), Ω in $\mathcal{L}a(H^{2n})$, induces the Cartan connection $\overline{\Omega}$ in $\mathcal{A}a(H^{2n})$;

see the isotropic case in Popov (1975), Popov and Daikhin (1975, 1976), and Bishop and Crittenden (1965). This is the unique connection on $\mathcal{A}a(H^{2n})$ represented as $i^*\overline{\Omega} = (\Omega, \chi)$, where χ is the shifting form and i: $\mathcal{A}a \to \mathcal{L}a$ is the trivial reduction of bundles. If $s_{\mathcal{U}}^{(a)}$ is a local adapted frame in $\mathcal{L}a(H^{2n})$, then $\bar{s}_{\mathcal{U}}^{(0)} = i \circ s_{\mathcal{U}}$ is a local section in $\mathcal{A}a(H^{2n})$ and

$$(\overline{\Omega}_{\mathcal{U}}) = s_{\mathcal{U}}\Omega = (\Omega_{\mathcal{U}}, \chi_{\mathcal{U}}) \tag{41}$$

$$(\overline{\mathcal{R}}_{\mathcal{U}}) = s_{\mathcal{U}}\overline{\mathcal{R}} = (\mathcal{R}_{\mathcal{U}}, T_{\mathcal{U}}) \tag{42}$$

where $\chi = e_{\hat{\alpha}} \otimes \chi^{\hat{\alpha}}{}_{\mu}X^{\mu}$, $G_{\alpha\beta} = \chi^{\hat{\alpha}}{}_{\alpha}\chi^{\hat{\beta}}{}_{\beta}\eta_{\hat{\alpha}\hat{\beta}}$ ($\eta_{\hat{\alpha}\hat{\beta}}$ is diagonal with $\eta_{\hat{\alpha}\hat{\alpha}} = \pm 1$) is a frame decomposition of metric (7) on H^{2n}, $e_{\hat{\alpha}}$ is the standard distinguished basis on $\mathbb{R}^{n+n}$, and the projection of torsion, $T_{\mathcal{U}}$, on base H^{2n} is defined as

$$\begin{aligned} T_{\mathcal{U}} &= d\chi_{\mathcal{U}} + \Omega_{\mathcal{U}} \wedge \chi_{\mathcal{U}} + \chi_{\mathcal{U}} \wedge \Omega_{\mathcal{U}} \\ &= e_{\hat{\alpha}} \otimes \sum_{\mu \subset \nu} T^{\hat{\alpha}}{}_{\mu\nu}X^{\mu} \wedge X^{\nu} \end{aligned} \tag{43}$$

For a fixed local adapted basis on $\mathcal{U} \subset H^{2n}$ we can identify components $T^{\hat{\alpha}}{}_{\mu\nu}$ of torsion (43) with components of torsion (11) on H^{2n}, i.e., $T^{\hat{\alpha}}{}_{\mu\nu} = \overset{N}{T}{}^{\alpha}{}_{\mu\nu}$. By straightforward calculation we obtain

$$(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = [(\Delta\mathcal{R})_{\mathcal{U}}, (R\tau)_{\mathcal{U}} + (Ri)_{\mathcal{U}}] \tag{44}$$

where

$$(R_{\tau})_{\mathcal{U}} = \hat{\delta}_G T_{\mathcal{U}} + *_G^{-1}[\Omega_{\mathcal{U}}, *_G T_{\mathcal{U}}]$$

$$(R_i)_{\mathcal{U}} = *_G^{-1}[\chi_{\mathcal{U}}, *_G \mathcal{R}_{\mathcal{U}}]$$

Form $(R_i)_{\mathcal{U}}$ from (44) is locally constructed by using components of the Ricci tensor [see (13)] as follows from decomposition on the local adapted basis $X^{\mu} = \delta u^{\mu}$:

$$(Ri)_{\mathcal{U}} = e_{\hat{\alpha}} \otimes (-1)^{2n+1}\overset{N}{R}_{\lambda\mu}G^{\hat{\alpha}\lambda}\delta u^{\mu} \tag{45}$$

We remark that for isotopic torsionless pseudo-Riemannian spaces the requirement that $(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = 0$, i.e., imposing the connection (38) to satisfy Yang–Mills equations (26) [equivalently, (32) or (34)], we obtain (Aldovandi and Stedile, 1984; Popov, 1975; Popov and Daikhin, 1975, 1976) the equivalence of the mentioned gauge gravitational equations with the vacuum Einstein equations $R_{ij} = 0$. In the case of GL-spaces with arbitrary given torsion, even considering vacuum gravitational fields, we have to introduce a source for gauge gravitational equations in order to compensate for the contribution of torsion and to obtain equivalence with the Einstein equations.

5.3. A Gaugelike Form of Gravitational Field Equations on H^{2n}

Considerations presented in Sections 5.1 and 5.2 constitute the proof of the following result.

Theorem. The Einstein equations (13) for the almost Hermitian model, H^{2n}, of the generalized Lagrange space M^n are equivalent to Yang–Mills equations

$$(\Delta\overline{\mathcal{R}}) = \overline{\mathcal{T}} \tag{46}$$

for the induced Cartan connection $\overline{\Omega}$ [see (38), (41)] in the bundle of local adapted affine frames $\mathcal{A}a(H^{2n})$ with source $\overline{\mathcal{T}}_{\mathcal{U}}$ constructed locally by using the same formulas (44) [for $(\Delta\overline{\mathcal{R}})_{\mathcal{U}}$], where $\overset{N}{R}_{\alpha\beta}$ is changed by the matter source $\overset{N}{T}_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}T$.

6. NONLINEAR DE SITTER GAUGE GRAVITY WITH LOCAL ANISOTROPY

The equivalent reexpression of the Einstein theory as a gaugelike theory implies, for both locally isotropic and anisotropic space-times, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames. A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group $\mathcal{A}f_{n+n}(\mathbb{R})$ to the de Sitter gauge group $S_{n+n} = SO(n + n + 1)$ acting on distinguished $\mathbb{R}^{n+n+1}$ space.

6.1. Nonlinear Gauge Theories of de Sitter Group

Let us consider the de Sitter space Σ^{2n} as a hypersurface given by the equations $\eta_{AB}u^A u^B = -l^2$ in the flat $2n$-dimensional spaces enabled with diagonal metric η_{AB}, $\eta_{AA} = \pm 1$ ($A, B, C, \ldots = 1, 2, \ldots, 2n + 1$), where $\{u^A\}$ are global Cartesian coordinates in $\mathbb{R}^{2n+1}$; $l > 0$ is the curvature of de Sitter space. The de Sitter group $S_{(\eta)} = SO_{(\eta)}(2n + 1)$ is defined as the isometry group of Σ^{2n}-space with $n(2n + 1)$ generators of Lie algebra $\mathcal{s}O_{(\eta)}(2n + 1)$ satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC} \tag{47}$$

Decomposing indices $A, B, \ldots$ as $A = (\hat{\alpha}, 2n + 1)$, $B = (\hat{\beta}, 2n + 1)$, $\ldots$, the metric η_{AB} as $\eta_{AB} = (\eta_{\hat{\alpha}\hat{\beta}}, \eta_{(2n+1)(2n+1)})$, and operators M_{AB} as $M_{\hat{\alpha}\hat{\beta}} = \mathcal{I}_{\hat{\alpha}\hat{\beta}}$ and $P_{\hat{\alpha}} = l^{-1}M_{2\hat{n}+1,\hat{\alpha}}$, we can write (47) as

$$[\mathcal{I}_{\hat{\alpha}\hat{\beta}}, \mathcal{I}_{\hat{\gamma}\hat{\delta}}] = \eta_{\hat{\alpha}\hat{\gamma}}\mathcal{I}_{\hat{\beta}\hat{\delta}} - \eta_{\hat{\beta}\hat{\gamma}}\mathcal{I}_{\hat{\alpha}\hat{\delta}} + \eta_{\hat{\beta}\hat{\delta}}\mathcal{I}_{\hat{\alpha}\hat{\gamma}} - \eta_{\hat{\alpha}\hat{\delta}}\mathcal{I}_{\hat{\beta}\hat{\gamma}}$$

$$[P_{\hat{\alpha}}, P_{\hat{\beta}}] = -l^{-2}\mathcal{I}_{\hat{\alpha}\hat{\beta}}, \qquad [P_{\hat{\alpha}}, \mathcal{I}_{\hat{\beta}\hat{\gamma}}] = \eta_{\hat{\alpha}\hat{\beta}}P_{\hat{\gamma}} - \eta_{\hat{\alpha}\hat{\gamma}}P_{\hat{\beta}}$$

where we have indicated the possibility to decompose $\mathcal{J}O_{(\eta)}(2n+1)$ into a direct sum, $\mathcal{J}O_{(\eta)}(2n+1) = \mathcal{J}O_{(\hat{\eta})}(2n) \oplus V_{2n}$, where V_{2n} is the vector space stretched on vectors $P_{\hat{A}}$. We remark that $\Sigma^{2n} = S_{(\eta)}/L_{(\hat{\eta})}$, where $L_{(\hat{\eta})} = SO_{(\hat{\eta})}(2n)$. For $\eta_{AB} = \text{diag}(1, -1, -1, -1)$ and $S_{10} = SO(1, 4)$, $SO(1, 3) = L_6$ is the group of Lorentz rotations.

Let $E(H^{2n}, \mathbb{R}^{2n+1}, S_{(\eta)}, P)$ be the vector bundle associated with principal bundle $P(S_{(\eta)}, H^{2n})$ on H^{2n}-spaces. The action of the structural group $S_{(\eta)}$ on E can be realized by using $(2n+1) \times (2n+1)$ matrices with a parametrization distinguishing subgroup $L_{(\hat{\eta})}$:

$$B = bB_L \tag{48}$$

where

$$B_L = \begin{vmatrix} L & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & 1 \end{vmatrix}$$

$L \in L_{(\hat{\eta})}$ is the de Sitter bust matrix transforming the vector $(0, 0, \ldots, \rho)$ $\in \mathbb{R}^{2n+1}$ into the arbitrary point $(V^1, V^2, \ldots, V^{2n+1}) \in \Sigma_\rho^{2n} \subset \mathbb{R}^{2n+1}$ with curvature $\rho(V_A V^A = -\rho^2, V^A = t^A \rho)$. Matrix b can be expressed as

$$b = \begin{bmatrix} \delta^{\hat{\alpha}}{}_{\hat{\beta}} + \dfrac{t^{\hat{\alpha}} t_{\hat{\beta}}}{(1 + t^{2n+1})} & \vdots & t^{\hat{\alpha}} \\ \cdots\cdots & & \cdots\cdots \\ t_{\hat{\beta}} & \vdots & t^{2n+1} \end{bmatrix}$$

The de Sitter gauge field is associated with a linear connection in E, i.e., with a $\mathcal{J}O_{(\eta)}(2n+1)$-valued connection 1-form on H^{2n}:

$$\tilde{\Omega} = \begin{pmatrix} \omega^{\hat{\alpha}}{}_{\hat{\beta}} & \tilde{\theta}^{\hat{\alpha}} \\ \tilde{\theta}_{\hat{\beta}} & 0 \end{pmatrix} \tag{49}$$

where $\omega^{\hat{\alpha}}{}_{\hat{\beta}} \in \mathcal{J}O_{(\eta)}(2n)$, $\theta^{\hat{\alpha}} \in \mathbb{R}^{2n}$, $\theta_{\hat{\beta}} = \eta_{\hat{\beta}\hat{\alpha}}\theta^{\hat{\alpha}}$.

Because $S_{(\eta)}$-transforms mix $\omega^{\hat{\alpha}}{}_{\hat{\beta}}$ and $\theta^{\hat{\alpha}}$ fields in (49) [the introduced parametrization is invariant on action on $SO_{(\hat{\eta})}(2n)$ group] we cannot identify $\omega^{\hat{\alpha}}{}_{\hat{\beta}}$ and $\theta^{\hat{\alpha}}$, respectively, with the connection $\overset{N}{\Gamma}{}^{\alpha}{}_{\beta\gamma}$ and the fundamental form χ^{α} in H^{2n} [as we have for (38) and (41)]. To avoid this difficulty we consider (Tseytlin, 1982; Ponomarev *et al.*, 1985) a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$, namely, we introduce into consideration the nonlinear gauge field

$$\Omega = b^{-1}\Omega b + b^{-1}db = \begin{vmatrix} \Gamma^{\hat{\alpha}}{}_{\hat{\beta}} & \theta^{\hat{\alpha}} \\ \theta_{\hat{\beta}} & 0 \end{vmatrix} \tag{50}$$

where

$$\Gamma^{\hat{\alpha}}{}_{\hat{\beta}} = \omega^{\hat{\alpha}}{}_{\hat{\beta}} - (t^{\hat{\alpha}} Dt_{\hat{\beta}} - t_{\hat{\beta}} Dt^{\hat{\alpha}})/(1 + t^{2n+1})$$

$$\theta^{\hat{\alpha}} = t^{2n+1}\tilde{\theta}^{\hat{\alpha}} + Dt^{\hat{\alpha}} - t^{\hat{\alpha}}(dt^{2n+1} + \tilde{\theta}_{\hat{\gamma}} t^{\hat{\gamma}})/(1 + t^{2n+1})$$

$$Dt^{\hat{\alpha}} = dt^{\hat{\alpha}} + \omega^{\hat{\alpha}}{}_{\hat{\beta}} t^{\hat{\alpha}}$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms $\Gamma' = L'\Gamma(L')^{-1} + L'd(L')^{-1}$, $\theta' = L\theta$, where the nonlinear matrix-valued function $L' = L'(t^{\alpha}, b, B_T)$ is defined from $B_b = b' B_{L'}$ [see parametrization (48)].

Now, we can identify components of (50) with components of $\overset{N}{\Gamma}{}^{\alpha}{}_{\beta\gamma}$ and $\chi^{\hat{\alpha}}{}_{\alpha}$ on H^{2n} and induce in a consistent manner on the base of bundle $E(P, \mathbb{R}^{2n}, S(\eta), H^{2n})$ the almost Hermitian Lagrange geometry.

6.2. Dynamics of the Nonlinear $S(\eta)$-Gravity with Local Anisotropy

Instead of the gravitational potential (38), we introduce the nonlinear gravitational connection [similar to (50)]

$$\Gamma = \begin{vmatrix} \Gamma^{\hat{\alpha}}{}_{\hat{\beta}} & l_0^{-1}\chi^{\hat{\alpha}} \\ l_0^{-1}\chi_{\hat{\beta}} & 0 \end{vmatrix} \tag{51}$$

where $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}} = \Gamma^{\hat{\alpha}}{}_{\hat{\beta}\mu}\delta u^{\mu}$, $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\mu} = \chi^{\hat{\alpha}}{}_{\alpha}\chi^{\beta}{}_{\hat{\beta}}\overset{N}{\Gamma}{}^{\alpha}{}_{\beta\mu} + \chi^{\hat{\alpha}}_{\sigma}\delta_{\mu}\chi^{\sigma}_{\hat{\beta}}$, $\chi^{\hat{\alpha}} = \chi^{\hat{\alpha}}{}_{\mu}\delta u^{\mu}$, and $G_{\alpha\beta} = \chi^{\hat{\alpha}}{}_{\alpha}\chi^{\hat{\beta}}{}_{\beta}\eta_{\hat{\alpha}\hat{\beta}}$, and $\eta_{\hat{\alpha}\hat{\beta}}$ is parametrized as

$$\eta_{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} \eta_{ij} & 0 \\ 0 & \eta_{(i)(j)} \end{pmatrix}, \qquad \eta_{ij} = \eta_{(i)(j)} = \mathrm{diag}(1, -1, \ldots, -1)$$

l_0 is a dimensional constant.

The curvature of (51), $\mathcal{R} = d\Gamma + \Gamma \wedge \Gamma$, can be written as

$$\mathcal{R} = \begin{bmatrix} \mathcal{R}^{\hat{\alpha}}{}_{\hat{\beta}} + l_0^{-1}\pi^{\hat{\alpha}}_{\hat{\beta}} & \vdots & l_0^{-1}T^{\hat{\alpha}} \\ \cdots\cdots & & \cdots\ \cdots \\ l_0^{-1}T^{\hat{\beta}} & \vdots & 0 \end{bmatrix} \tag{52}$$

where $\pi^{\hat{\alpha}}{}_{\hat{\beta}} = \chi^{\hat{\alpha}} \wedge \chi_{\hat{\beta}}$, $\mathcal{R}^{\hat{\alpha}}{}_{\hat{\beta}} = \frac{1}{2}\mathcal{R}_{\hat{\beta}}{}^{\hat{\alpha}}{}_{\mu\nu}\delta u^{\mu} \wedge \delta u^{\nu}$, and $\mathcal{R}_{\hat{\beta}}{}^{\hat{\alpha}}{}_{\mu\nu} = \chi_{\hat{\beta}}{}^{\beta}\chi_{\alpha}{}^{\hat{\alpha}}\overset{N}{R}{}_{\beta\cdot\mu\nu}^{\cdot\alpha}$ [see (12), the components of H^{2n}-curvature]. The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational locally anisotropic theory [bundle metric (23) for $S_{(\eta)}$-group is nondegenerate]. The Lagrangian of the theory is postulated as

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined as

$$L_{(G)} = \frac{1}{4\pi} \operatorname{Tr}(\mathcal{R} \wedge *_G \mathcal{R}) = \mathcal{L}_{(G)} |G|^{1/2} \delta^{2n} u$$

$$\mathcal{L}_{(G)} = \frac{1}{2l^2} T^{\hat{\alpha}}{}_{\mu\nu} T_{\hat{\alpha}}{}^{\mu\nu} + \frac{1}{8\lambda} \mathcal{R}_{\hat{\beta}}{}^{\hat{\alpha}}{}_{\mu\nu} \mathcal{R}_{\hat{\alpha}}{}^{\hat{\beta}\mu\nu} - \frac{1}{l^2} (R(\Gamma) - 2\lambda_1) \quad (53)$$

$T^{\hat{\alpha}}{}_{\mu\nu} = \chi^{\hat{\alpha}}{}_{\alpha} \overset{N}{T}{}^{\alpha}{}_{\mu\nu}$ [the gravitational constant l^2 in (53) satisfies the relations $l^2 = 2l_0{}^2\lambda$, $\lambda_1 = -3/l_0$], Tr denotes the trace on $\hat{\alpha}$, $\hat{\beta}$ indices, and the matter field Lagrangian is defined as

$$L_{(m)} = -\frac{1}{2} \operatorname{Tr}(\Gamma \wedge *_G \mathcal{J}) = \mathcal{L}_{(m)} |G|^{1/2} \delta^8 u$$

$$\mathcal{L}_{(m)} = \frac{1}{2} \Gamma^{\hat{\alpha}}{}_{\hat{\beta}\mu} S^{\hat{\beta}}{}_{\alpha}{}^{\mu} - t^{\mu}{}_{\hat{\alpha}} h^{\hat{\alpha}}{}_{\mu} \quad (54)$$

The matter field source $\mathcal{J}$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on Γ and is parametrized as

$$\mathcal{J} = \begin{vmatrix} S^{\hat{\alpha}}{}_{\hat{\beta}} & -l_0 t^{\hat{\alpha}} \\ -l_0 t_{\hat{\beta}} & 0 \end{vmatrix} \quad (55)$$

with $t^{\hat{\alpha}} = t^{\hat{\alpha}}{}_{\mu} \delta u^{\mu}$ and $S^{\hat{\alpha}}{}_{\hat{\beta}} = S^{\hat{\alpha}}{}_{\hat{\beta}\mu} \delta u^{\mu}$ being respectively the canonical tensors of energy-momentum and spin density. Because of the contraction of the "interior" indices $\hat{\alpha}$, $\hat{\beta}$, in (53) and (54) we used the Hodge operator $*_G$ instead of $*_h$ (hereafter we consider $*_G = *$).

Varying the action

$$S = \int |G|^{1/2} \delta^{2n} u \, (\mathcal{L}_{(G)} + \mathcal{L}_{(m)})$$

on the Γ-variables (51), we obtain the gauge-gravitational field equations:

$$d(*\mathcal{R}) + \Gamma \wedge (*\mathcal{R}) - (*\mathcal{R}) \wedge \Gamma = -\lambda(*\mathcal{J}) \quad (56)$$

Specifying the variations on $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}}$ and $h^{\hat{\alpha}}$-variables, we rewrite (56) as

$$\hat{\mathcal{D}}(*\mathcal{R}) + \frac{2\lambda}{l^2} (\hat{\mathcal{D}}(*\pi) + \chi \wedge (*T^T) - (*T) \wedge \chi^T = -\lambda * S \quad (57)$$

$$\mathcal{D}(*T) - (*\mathcal{R}) \wedge \chi - \frac{2\lambda}{l^2} (*\pi) \wedge \chi = \frac{l^2}{2} \left(*t + \frac{1}{\lambda} * \tau \right) \quad (58)$$

where

$$T^T = \{T_{\hat{\alpha}}, T_{\hat{\alpha}} = \eta_{\hat{\alpha}\hat{\beta}} T^{\hat{\beta}}, T^{\hat{\beta}} = \frac{1}{2} T^{\hat{\beta}}{}_{\mu\nu} \delta u^{\mu} \wedge \delta u^{\nu}\}$$

$$\chi^T = \{\chi_{\hat{\alpha}}; \chi_{\hat{\alpha}} = \eta_{\hat{\alpha}\hat{\beta}} \chi^{\hat{\beta}}, \chi^{\hat{\beta}} = \chi^{\hat{\beta}}{}_{\mu} \delta u^{\mu}\}, \qquad \hat{\mathcal{D}} = d + \hat{\Gamma}$$

($\hat{\Gamma}$ acts as $\Gamma^{\hat{\alpha}}{}_{\hat{\beta}\mu}$ on indices $\hat{\gamma}$, $\hat{\delta}$, . . . and as $\hat{\Gamma}^{\alpha}{}_{\beta\mu}$ on indices γ, δ, . . .). In (58), τ defines the energy-momentum tensor of the $S_{(\eta)}$-gauge gravitational field $\hat{\Gamma}$:

$$\tau_{\mu\nu}(\hat{\Gamma}) = \frac{1}{2}\,\mathrm{Tr}\left(\mathcal{R}_{\mu\nu}\mathcal{R}^{\alpha}_{\nu} - \frac{1}{4}\,\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}G_{\mu\nu}\right) \tag{59}$$

Equations (56) [or equivalently (57), (58)] make up the complete system of variational field equations for nonlinear de Sitter gauge gravity with local anisotropy. They can be interpreted as a generalization of Miron's equations (13) for H^{2n}-gravity [equivalently, of gauge gravitational equations (46)] to a system of gauge field equations with dynamical torsion and corresponding spin-density source.

Tseytlin (1982) presents a quantum analysis of the isotropic version of equations (57) and (58). Of course, the problem of quantizing gravitational interactions is unsolved for both variants of locally anisotropic and isotropic gauge de Sitter gravitational theories, but we think that the generalized Lagrange version of $S_{(\eta)}$-gravity is more adequate for studying quantum radiational and statistical gravitational processes. This is a matter for further investigations.

Finally, we remark that we can obtain a nonvariational Poincaré gauge gravitational theory on GL-spaces if we consider the contraction of the gauge potential (51) to a potential with values in the Poincaré Lie algebra

$$\Gamma = \begin{vmatrix} \Gamma^{\hat{\alpha}}{}_{\hat{\beta}} & l_0^{-1}\chi^{\hat{\alpha}} \\ l_0^{-1}\chi_{\hat{\beta}} & 0 \end{vmatrix} \rightarrow \Gamma = \begin{vmatrix} \Gamma^{\hat{\alpha}}{}_{\hat{\beta}} & l_0^{-1}\chi^{\hat{\alpha}} \\ 0 & 0 \end{vmatrix}$$

Isotropic Poincaré gauge gravitational theories are studied in a number of papers (see, for example, Walner, 1985; Tseytlin, 1982; Luehr and Rosenbaum, 1980; Ponomarev *et al.*, 1985; Aldovandi and Stedile, 1984). In a manner similar to considerations presented in this work, we can generalize Poincaré gauge models for spaces with local anisotropy.

7. GRAVITATIONAL GAUGE INSTANTONS WITH LOCAL ANISOTROPY

The existence of self-dual, or instanton, topologically nontrivial solutions of Yang–Mills equations is a very important physical consequence of gauge theories. All known instanton-type Yang–Mills and gauge gravitational solutions (Tseytlin, 1982; Ponomarev *et al.*, 1985) are locally isotropic. A variational gauge-gravitational extension of Miron H^{2n}-gravity makes possible a straightforward application of techniques of constructing solutions for first-order gauge equations for the definition of locally anisotropic gravitational instantons. This section is devoted to the study of some particular instanton solutions of H^{2n}-gauge gravitational theory on GL-space.

Let us consider the Euclidean formulation of the $S_{(\eta)}$-gauge gravitational theory by changing gauge structural groups and flat metric:

$$SO_{(\hat{\eta})}(2n+1) \to SO(2n+1), \qquad SO_{(\hat{\eta})}(2n) \to SO(2n), \qquad \eta_{AB} \to -\delta_{AB}$$

Self-dual (anti-self-dual) conditions for the curvature (52)

$$\mathcal{R} = *\mathcal{R} \quad (-*\mathcal{R}) \tag{60}$$

can be written as a system of equations

$$(\mathcal{R}^{\hat{\alpha}}{}_{\hat{\beta}} - l_0^{-2}\pi^{\hat{\alpha}}{}_{\hat{\beta}}) = \pm*(\mathcal{R}^{\hat{\alpha}}{}_{\hat{\beta}} - l_0^{-2}\pi^{\hat{\alpha}}{}_{\hat{\beta}}) \tag{61}$$

$$T^{\hat{\alpha}} = \pm*T^{\hat{\alpha}} \tag{62}$$

(the "$-$" refers to the anti-self-dual case), where the "$-$" before l_0^{-2} appears because of the transition of the Euclidean negatively defined metric $-\delta_{\hat{\alpha}\hat{\beta}}$, which leads to $\chi^{\hat{\alpha}}{}_{\alpha} \to i\chi^{\hat{\alpha}}{}_{\alpha|E}$, $\pi \to -\pi_E$ (we shall omit the index E for Euclidean values).

For solutions of (61) and (62) the energy-momentum tensor (59) is identically equal to zero. Vacuum equations (56) and (57), when the source is $\mathcal{J} \equiv 0$ [see (55)], are satisfied as a consequence of generalized Bianchi identities for the curvature (52). The mentioned solutions of (61) and (62) realize a local minimum of the Euclidean action

$$S = \frac{1}{8\lambda}\int |G|^{1/2}\delta^{2n}u\{(R(\Gamma) - l_0^{-2}\pi)^2 + 2T^2\} \tag{63}$$

where $T^2 = T^{\hat{\alpha}}{}_{\mu\nu}T\hat{\alpha}^{\mu\nu}$ is extremal on the topological invariant (Pontryagin index)

$$p_2 = -\frac{1}{8\pi^2}\int \mathrm{Tr}(\mathcal{R} \wedge \mathcal{R}) = -\frac{1}{8\pi^2}\int \mathrm{Tr}(\hat{\mathcal{R}} \wedge \hat{\mathcal{R}})$$

For the Euclidean de Sitter spaces, when

$$\mathcal{R} = 0 \quad \left\{T = 0,\ R^{\hat{\alpha}\hat{\beta}}{}_{\mu\nu} = -\frac{2}{l_0^2}\chi^{[\alpha}{}_{\mu}\chi^{\beta]}{}_{\nu}\right\} \tag{64}$$

we obtain the absolute minimum, $S = 0$.

We emphasize that for $R_{\beta}{}^{\alpha}{}_{\mu\nu} = (2/l_0^2)\delta^{\alpha}_{[\mu}G_{\nu]\beta}$ torsion vanishes. Torsionless instantons also have another interpretation. For $T^{\alpha}{}_{\beta\gamma} = 0$ contraction of equations (61) leads to Einstein equations with cosmological λ-term (as a consequence of generalized Ricci identities):

$$\overset{N}{R}_{\alpha\beta\mu\nu} - \overset{N}{R}_{\mu\nu\alpha\beta} = \frac{3}{2}\{\overset{N}{R}_{[\alpha\beta\mu]\nu} - \overset{N}{R}_{[\alpha\beta\nu]\mu} + \overset{N}{R}_{[\mu\nu\alpha]\beta} - \overset{N}{R}_{[\mu\nu\beta]\alpha}\}$$

So, in the Euclidean case the locally anisotropic vacuum Einstein equations are a subset of instanton solutions.

Now, let us study the $SO(n)$ solution of equations (61) and (62). We consider the spherically symmetric ansatz

$$\Gamma^{\hat\alpha}{}_{\hat\beta\mu} = a(u)(u^{\hat\alpha}\delta_{\hat\beta\mu} - u_{\hat\beta}\delta^{\hat\alpha}_{\mu}) + q(u)\epsilon^{\hat\alpha}{}_{\hat\beta\mu\nu}u^{\nu}$$
$$\chi^{\hat\alpha}_{\alpha} = f(u)\delta^{\hat\alpha}{}_{\alpha} + n(u)u^{\hat\alpha}u_{\alpha}, \qquad N^{\hat\imath}_{j}(u) \equiv 0, \tag{65}$$

where $u^2 = X^{\hat\alpha}_{\alpha}\, u^{\alpha}$, $u = u^{\alpha}u^{\beta}\, G_{\alpha\beta} = x^{\hat\imath}x_{\hat\imath} + y^{\hat\jmath}j_{\hat\jmath}$, and $a(u)$, $q(u)$, $f(u)$ and $n(u)$ are some scalar functions. Introducing (65) into (61) and (62), we obtain, respectively,

$$u\left(\pm\frac{dq}{du} - a^2 - q^2\right) + 2(a \pm q) + (l_0)^{-1}f^2 = 0 \tag{66}$$

$$2d(a \mp q)/du + (a \mp q)^2 - l_0^{-1}fn = 0 \tag{67}$$

$$2\frac{df}{du} + f(a \mp 2q) + n(au - 1) = 0 \tag{68}$$

The traceless part of the torsion vanishes because of the parametrization (65), but in the general case the trace and pseudotrace of the torsion are not identical to zero:

$$T^{\mu} = q^{(0)}u^{\mu}(-2\, df/du + n - a(f + un))$$
$$\check{T}^{\mu} = q^{(1)}u^{\mu}(2qf)$$

$q^{(0)}$ and $q^{(1)}$ are constant. Equation (62) or (68) establishes the proportionality of T^{μ} and $\check{T}^{\mu}$. As a consequence we obtain that the $SO(2n)$ solution of (65) is torsionless if $q(u) = 0$ or $f(u) = 0$.

Let first analyze the torsionless instantons, $T^{\mu}{}_{\alpha\beta} = 0$. If $f = 0$, then from (67) one has two possibilities: (a) $n = 0$ leads to nonsense because $\chi^{\hat\alpha}_{\alpha} = 0$ or $G_{\alpha\beta} = 0$. (b) $a = u^{-1}$ and $n(u)$ is an arbitrary scalar function; we have from (67) $a \mp q = 2/(a + C^2)$ or $q = \pm 2/u(u + C^2)$, where $C = \text{const}$. If $q(u) = 0$, we obtain the de Sitter space (64) because equations (66) and (67) impose vanishing of both self-dual and anti-self-dual parts of $(\mathcal{R}^{\hat\alpha}{}_{\hat\beta}$ −

$l_0^2\pi^{\hat\alpha}{}_{\hat\beta}$), so, as a consequence, $\mathcal{R}^{\hat\alpha}{}_{\hat\beta} - l_0^2\pi^{\hat\alpha}{}_{\hat\beta} \equiv 0$. There is an infinite number of $SO(2n)$-symmetric solutions of (64):

$$f = l_0[a(2 - au)]^{1/2}, \qquad n = l_0\left\{2\frac{da}{du} + \frac{a^2}{[a(2 - au)]^{1/2}}\right\}$$

$a(u)$ is a scalar function.

To find instantons with torsion, $T^{\alpha}{}_{\beta\gamma} \neq 0$, is also possible. We present the $SO(4)$ one-instanton solution, obtained in Ponomarev *et al.*(1985) [which in our anisotropic case can be rewritten for H^4-space parametrized by local coordinates $(x^{\hat1}, x^{\hat2}, y^{\hat1}, y^{\hat2})$, with $u = x^{\hat1}x_{\hat1} + x^{\hat2}x_{\hat2} + y^{\hat1}y_{\hat1} + y^{\hat2}y_{\hat2}$]:

$$a = a_0(u + c^2)^{-1}, \qquad q = \mp q_0(u + c^2)^{-1}$$

$$f = l_0(\alpha u + \beta)^{1/2}/(u + c^2), \qquad n = c_0/(u + c^2)(\gamma u + \delta)^{1/2}$$

where

$$a_0 = -1/18, \qquad q_0 = 5/6, \qquad \alpha = 266/81, \qquad \beta = 8/9$$

$$\gamma = 10773/11858, \qquad \delta = 1458/5929$$

We suggest that local regions with $T^{\alpha}_{\beta\gamma} \neq 0$ are similar to Abrikosov vortexes in superconductivity and the appearance of torsion is a possible realization of the Meisner effect in gravity [for details and discussions on the superconducting or Higgs-like interpretation of gravity see Tseytlin (1982) and Ponomarev *et al.* (1985)]. In our case we obtain a locally anisotropic superconductivity and we think that the formalism of gauge locally anisotropic theories may even work for some models of anisotropic low- and high-temperature superconductivity (Vacaru, 1991).

8. OUTLOOK AND CONCLUSIONS

In this paper we have reformulated the fiber bundle formalism for both Yang–Mills and gravitational fields in order to include into consideration space-times with local anisotropy. We have argued that our approach has the advantage of making manifest the relevant structures of the theories with local anisotropy and putting greater emphasis on the analogy with isotropic models than the standard coordinate formulations in Finsler geometry.

Our geometrical approach to locally anisotropic gauge and gravitational interactions are refined in such a way as to shed light on some of the more specific properties and common and distinguishing features of the Yang–Mills and Einstein fields. As we have shown, it is possible to make a gaugelike treatment for both models with local anisotropy (by using correspondingly defined linear connections in bundle spaces with semisimple structural groups, with variants of nonlinear realization and extension to semisimple structural groups, for gravitational fields).

We have proposed a gauge interpretation of locally anisotropic gravity starting from the almost Hermitian model of generalized Lagrange spaces. This construction is more general than that based on gauge fields in the Finsler bundle on space-time (Asanov and Ponomarenko, 1989) (see Remark 1 in this paper) and differs essentially from other Finsler variants of gravity (Matsumoto, 1986; Asanov and Ponomarenko, 1989).

We note that there are a number of arguments for the necessity to take into account physical effects of possible local anisotropy. The first one is the well-known result that a self-consistent theoretical description of radiational processes in classical field theories is possible by adding high derivation terms (for example, electromagnetic radiation of accelerating charged particles in classical electrodynamics is modeled by introducing additional terms proportional to the third time derivative of the coordinates). The second, very important argument for investigations of quantum models on tangent bundles is the unclosed character of quantum electrodynamics; for values of momenta $p \rightarrow \infty$ the renormalized amplitudes of quantum electrodynamic processes also tend to ∞, which requires additional (less motivated from a physical point of view) suppositions and modifications of fundamental principles of the theory. We have to introduce similar, but more complicated, considerations in order to model gravitational radiational dissipation in all variants of classical and quantum gravity and quantum field theories with high derivatives. That is, a careful analysis of physical processes when the weak reaction of classical and quantum systems interacting or being measured is not negligibly small requires extensions of the geometrical background of the theories. Such generalizations seem to be more appropriate for describing statistical gravitational and gauge effects (Vlasov, 1966; Vacaru, 1995a) in classical and quantum field theory.

Of course, physical features of models with local anisotropy do not yet have strong experimental support [except for models of continuous media with dislocations and disclinations (Kadic and Edelen, 1983)]. The physical status of nonlinear connection [see the covariant derivation (2)] is still unclear, as is the physical interpretation of the energy-momentum tensor on H^{2n}-spaces [the source of H^{2n}-gravitational Einstein equations (13) and the manner of formulation of conservation laws on spaces with local anisotropy require additional analyses]. We hope that these questions will be solved in the framework of models with turbulent space-time, curved momentum, or phase spaces. The first step is to propose the corresponding geometrical formalism and to formulate the basic principles and field equations for fundamental interactions (in this work we have paid attention to gauge and gravitational fields).

Finally, we emphasize that we have developed new types of gauge and gravitational theories emerging from local anisotropic considerations and that further investigations in the direction of formulating classical and quantum field theory on generalized Lagrange spaces are in progress (Miron and Kawaguchi, 1991; Miron and Anastasiei, 1993; Vlasov, 1966; Miron, 1985; Vacaru, 1995a,b; Vacaru and Ostaf, 1993).

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